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A Mathematical Demonstration of the Pareto Optimality of Pay-As-You-Go Social Security Programs in a Closed Economy

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In a recent article, Breyer [1989] concludes that it is impossible to compensate pensioners in the transition from a pay-as-you-go public pension system to a privatized or funded system without making at least one later generation worse off; Breyer reaches this conclusion in the context of a simple overlapping generations model of a closed economy under the assumption that the transition results in increased saving by workers. Although this conclusion is correct under the increased saving assumption in the relevant domain of the production function, the proof that Breyer provides is not sufficient to establish that fact. This note extends Breyer's analysis to provide a sufficient proof.

To facilitate comparison, essentially the same model and notation used by Breyer [1989] is adopted here. A closed economy with no uncertainty is assumed, populated by two-period life cycle consumers who possess equal abilities and share identical utility functions. Individuals are assumed to work, pay taxes to a pay-as-you-go public pension program, consume, and save during the first period of the life cycle. In the second period of the life cycle, individuals are assumed to retire completely and consume the entire proceeds of their first period saving as well as their public pension benefit, leaving no bequest.

Workers at time t are referred to as cohort t, and their population is represented as

$$(1) \quad N_t = N_{t-1}(1+g_t) = N_{t-1}G_t \ ,$$

where g_t denotes the rate of growth in the working population relative to the last cohort. A neoclassical economy with a single good used for both consumption and capital is assumed, with a strictly concave, twice differentiable, constant returns to scale production function exhibiting positive but diminishing marginal products; specifically,

$$(2) \quad Y_t = F(K_t, N_t) \ ,$$

where Y_t and K_t represent output and the available capital stock, respectively, at time t. Capital is assumed to be completely consumed in the production process, so that the capital available at any time is equal to the savings of the previous working cohort, now retired; i.e.,

$$(3) \quad K_{t+1} = N_t s_t \Rightarrow k_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \left[\frac{N_t}{N_{t+1}}\right] s_t = \frac{s_t}{G_{t+1}} ,$$

where s_t denotes the savings per member of cohort t during their working period and k_t denotes capital per worker. Assuming that both factors are paid their marginal products, the constant returns to scale assumption ensures that factor payments exhaust output; i.e.,

$$(4) \quad Y_t = R_t K_t + w_t N_t \Rightarrow y_t = \frac{Y_t}{N_t} = R_t k_t + w_t ,$$

where R_t and w_t represent the marginal products of capital[1] and labor, respectively, and y_t represents output per worker.

The utility function shared by all consumers is assumed to be twice differentiable and strictly quasi-concave. Consumption per person in the first period of the life cycle can be represented as

$$(5) \quad c_t = w_t(1-\tau) - s_t ,$$

where τ represents the public pension tax rate, assumed constant over time.[2] Consumption per person in the second period of the life cycle for cohort t is represented as

$$(6) \quad z_{t+1} = R_{t+1} s_t + p_{t+1} ,$$

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[1]Because a unit depreciation rate is assumed, the rate of return to capital net of depreciation (r_t) is equal to the marginal product of capital less one; i.e., $r_t = R_t - 1$.

[2]Breyer denotes the public pension tax payment as b.

where p_{t+1} denotes the pay-as-you-go public pension benefit per retired member of cohort t. The level of public pension benefits can be derived from the assumption of a strict pay-as-you-go program; i.e.,

$$(7) \quad \tau w_t N_t = p_t N_{t-1} \;\Rightarrow\; p_t = \tau w_t G_t \;\Rightarrow\; z_{t+1} = R_{t+1} s_t + \tau w_{t+1} G_{t+1} \,.$$

Suppose now that we decide to abolish the public pension program and institute in its place a program of compensating transfers from the working to retired population each period. These transfers are designed to maintain the lifetime utility of each affected cohort, so that the abolition of the public pension program makes no cohort worse off. Clearly, the substitution of the compensating transfers program for the public pension program has changed nothing except in name–because the compensating transfers to the retired and associated taxes on the working will be identical to what the corresponding public pension benefits and taxes would have been, the substitution of the compensating transfers program leaves the lifetime wealth, and therefore consumption, of all cohorts unchanged.[3]

To make the problem more interesting, then, we ask the question whether it is possible to make at least one cohort better off without making any cohort worse off by instituting such a compensating transfers program while at the same time requiring each working cohort to save more for their own retirement; i.e., suppose we abolish the public pension program, force each

[3]An equivalent result is obtained if a pay-as-you-go public pension program is privatized by issuing new public debt to holders of pension rights under the public pension program and then instituting tax payments on workers such that the privatization debt grows at the same rate over time as would have the unfunded liability under the public pension program. In this sense, privatization plans can always be found, at least in theory, that have no real economic effects and leave the lifetime utility of all present and future cohorts unaffected. See Leimer [1991 (A) and 1991 (B)] for additional discussion of these issues.

working cohort to increase their saving, but then compensate them during their retirement period so that their lifetime utility is at least as great as under the public pension program. Does the required pattern of compensation implied by this plan diminish over time?

One suggestion encountered in the privatization debate, for example, is that privately available rates of return exceed the implicit rate of return from a mature public pension program by such a wide margin that, if allowed to save privately for their own retirement, workers might be able to fund their own retirement as well as honor the pension rights earned under the public pension program by preceding generations.[4] The question posed here is less demanding–can successive generations of workers, by saving more privately, fund the pension rights of the previous generation as well as at least part of their own retirement, so that the compensation required to maintain the lifetime utility of each cohort gradually diminishes over time?

In his analysis, Breyer [1989] showed that if such a compensation pattern exists, it is characterized by the property that the compensating payments made by each cohort always exceed what their tax payments would have been under the public pension program. By itself, however, this result does not imply, as suggested by Breyer, that feasible Pareto-equivalent compensation schemes do not exist.[5] Suppose, for example, that a monotonically-decreasing compensation pattern exists that asymptotically approaches the level of what tax payments would

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[4]For example, see Ferrara [1985]. Other analysts, including Buchanan [1979], have suggested that Pareto-superior privatization schemes may exist. In the context of a model with exogenously determined factor returns, Townley [1981] claims to identify a Pareto-superior scheme for gradually converting a dynamically inefficient public pension program to a fully funded basis. See Leimer [1991 (A)] for a general refutation of these claims.

[5]In this context, a feasible scheme is one that can be sustained by workers out of their wages.

have been under the public pension program. Such a compensation pattern would be feasible and Pareto-equivalent to the public pension program. Given this possibility, then, the analysis must be carried a bit farther to establish the properties of the required compensation pattern under such a program.

As in the Breyer analysis, variable values under the new compensating transfers program are denoted by the "^{o}" superscript. Consumption per member of cohort t during the working period, then, is given by

$$(8) \quad c_t^o = w_t^o - s_t^o - \varphi_t \ ,$$

where φ_t denotes the compensating transfer payment per member of cohort t made to the preceding cohort, now retired. Similarly, consumption per member of cohort t during the retirement period is given by

$$(9) \quad z_{t+1}^o = R_{t+1}^o s_t^o + p_{t+1}^o \ ,$$

where p_{t+1}^o denotes the transfer received per retired member of cohort t as compensation for the loss of its public pension benefit and any loss in lifetime utility caused by the forced increase in saving during the working period. Because the compensating transfers to each retired cohort are financed by a tax on the immediately succeeding cohort, it follows that

$$(10) \quad N_t p_{t+1}^o = N_{t+1} \varphi_{t+1} \ \Rightarrow \ p_{t+1}^o = G_{t+1} \varphi_{t+1} \ \Rightarrow \ z_{t+1}^o = R_{t+1}^o s_t^o + G_{t+1} \varphi_{t+1} \ .$$

Because the model assumes that labor supply is fixed within each period, the strict concavity of the production function ensures that the rate of return to capital falls as saving is increased under the compensating transfers program; i.e.,

$$(11) \quad s_t^o > s_t \Rightarrow R_{t+1}^o < R_{t+1} .$$

The strict concavity of the production function also implies that

$$(12) \quad R_{t+1}(k_{t+1}^o - k_{t+1}) > y_{t+1}^o - y_{t+1} .$$

Substituting from equations (3) and (4),

$$R_{t+1}(k_{t+1}^o - k_{t+1}) > R_{t+1}^o k_{t+1}^o + w_{t+1}^o - R_{t+1} k_{t+1} - w_{t+1} \Rightarrow$$

$$(13) \quad w_{t+1}^o - w_{t+1} < k_{t+1}^o (R_{t+1} - R_{t+1}^o) = s_t^o \left[\frac{R_{t+1} - R_{t+1}^o}{G_{t+1}} \right] .$$

Denote the first cohort required to save more under the compensating transfers program as cohort 1. Assume that this cohort makes a compensating payment to the preceding cohort, now retired, to compensate the members of that cohort for the loss of their public pension benefits. Members of the preceding cohort, then, are no better or worse off than under the public pension program. Compensating transfers must be made to members of cohort 1 during their retirement period, however, to adjust for the greater saving that they were required to do during their working period as well as for the loss of their public pension benefits; i.e.,

$$(14) \quad c_1^o = w_1 - \varphi_1 - s_1^o = w_1(1-\tau) - s_1^o , \qquad \text{and}$$

$$(15) \quad z_2^o = R_2^o s_1^o + G_2 \varphi_2 ,$$

as in equations (8) and (10). From equations (5), (14), and (11), it follows that working period consumption must fall for cohort 1 under the compensating transfers program; i.e.,

$$(16) \quad c_1 + s_1 = c_1^o + s_1^o \Rightarrow c_1 - c_1^o = s_1^o - s_1 > 0 .$$

In order to maintain lifetime utility for this cohort, then, retirement period consumption under the compensating transfers program must exceed that under the public pension program. In particular, the strict quasi-concavity of the utility function requires that

$$z_2^o - z_2 > R_2(c_1 - c_1^o) \Rightarrow R_2^o s_1^o + G_2\varphi_2 - R_2 s_1 - \tau w_2 G_2 > R_2(s_1^o - s_1)$$

$$\Rightarrow \quad s_1^o(R_2^o - R_2) + G_2(\varphi_2 - \tau w_2) > 0 \Rightarrow \varphi_2 - \tau w_2 > s_1^o(R_2 - R_2^o)/G_2 \Rightarrow$$

$$(17) \quad \varphi_2 > \tau w_2 + s_1^o(R_2 - R_2^o)/G_2 > 0 \ ,$$

in order for lifetime utility to be maintained for cohort 1 under the compensating transfers program.

The results for cohort 2 are similar, except that working period wages now differ between the two programs; i.e.,

$$(18) \quad c_2^o = w_2^o - s_2^o - \varphi_2 \ , \qquad \text{and}$$

$$(19) \quad z_3^o = R_3^o s_2^o + G_3\varphi_3 \ .$$

From equations (5), (18), (13), (11), and (17), we know that working period consumption for cohort 2 is less under the compensating transfers program than under the public pension program; i.e.,

$$(20) \quad c_2 - c_2^o = w_2(1-\tau) - s_2 - w_2^o + s_2^o + \varphi_2 = (s_2^o - s_2) + (\varphi_2 - \tau w_2) - (w_2^o - w_2) \Rightarrow$$

$$(21) \quad c_2 - c_2^o > (s_2^o - s_2) + \varphi_2 - \tau w_2 - s_1^o\left[\frac{R_2 - R_2^o}{G_2}\right] > 0 \ .$$

Consequently, maintaining lifetime utility for this cohort requires that consumption in the second period of the life cycle under the compensating transfers program must exceed that under the public pension program. In particular, the strict quasi-concavity of the utility function requires that

$$
\begin{aligned}
& z_3^o - z_3 > R_3 (c_2 - c_2^o) \\
\Rightarrow \quad & R_3^o s_2^o + G_3 \varphi_3 - R_3 s_2 - \tau w_3 G_3 > R_3 [s_2^o - s_2 + \varphi_2 - \tau w_2 - (w_2^o - w_2)] \\
\Rightarrow \quad & s_2^o (R_3^o - R_3) + G_3 (\varphi_3 - \tau w_3) > R_3 [\varphi_2 - \tau w_2 - (w_2^o - w_2)] \\
\Rightarrow \quad & \varphi_3 - \tau w_3 - s_2^o \left[\frac{R_3 - R_3^o}{G_3} \right] > \frac{R_3}{G_3} [\varphi_2 - \tau w_2 - (w_2^o - w_2)] \quad \Rightarrow
\end{aligned}
$$

$$
\varphi_3 - \tau w_3 - (w_3^o - w_3) > \frac{R_3}{G_3} [\varphi_2 - \tau w_2 - (w_2^o - w_2)] > 0 \; . \tag{22}
$$

To generalize equation (22) to subsequent periods, suppose that the relationship

$$
\varphi_t - \tau w_t - (w_t^o - w_t) > \frac{R_t}{G_t} [\varphi_{t-1} - \tau w_{t-1} - (w_{t-1}^o - w_{t-1})] > 0 \tag{23}
$$

holds for some $t \geq 3$. From equations (5), (8), (11), and (23), we know that

$$
c_t - c_t^o = w_t (1 - \tau) - s_t - w_t^o + s_t^o + \varphi_t = (s_t^o - s_t) + [\varphi_t - \tau w_t - (w_t^o - w_t)] > 0 \; . \tag{24}
$$

Because consumption for cohort t during the working period under the compensating transfers program falls below that under the public pension program, maintaining lifetime utility for this cohort requires that consumption in the retirement period exceed that under the public pension program. In particular, the strict quasi-concavity of the utility function requires that

$$z^o_{t+1} - z_{t+1} > R_{t+1}(c_t - c^o_t)$$

$$\Rightarrow \quad R^o_{t+1} s^o_t + G_{t+1}\varphi_{t+1} - R_{t+1} s_t - \tau w_{t+1} G_{t+1} > R_{t+1}[s^o_t - s_t + \varphi_t - \tau w_t - (w^o_t - w_t)]$$

$$\Rightarrow \quad s^o_t (R^o_{t+1} - R_{t+1}) + G_{t+1}(\varphi_{t+1} - \tau w_{t+1}) > R_{t+1}[\varphi_t - \tau w_t - (w^o_t - w_t)]$$

$$\Rightarrow \quad \varphi_{t+1} - \tau w_{t+1} - s^o_t \left[\frac{R_{t+1} - R^o_{t+1}}{G_{t+1}}\right] > \frac{R_{t+1}}{G_{t+1}}[\varphi_t - \tau w_t - (w^o_t - w_t)] \quad \Rightarrow$$

$$\text{(25)} \quad \varphi_{t+1} - \tau w_{t+1} - (w^o_{t+1} - w_{t+1}) > \frac{R_{t+1}}{G_{t+1}}[\varphi_t - \tau w_t - (w^o_t - w_t)] > 0 .$$

Because equation (22) demonstrates that equation (23) holds for $t = 3$, it follows that equation (25) must hold for all $t \geq 3$.

To develop the implications of this result, let

$$\text{(26)} \quad \delta_t = \varphi_t - \tau w_t - (w^o_t - w_t) ;$$

i.e., δ_t represents the amount by which the compensating payment made by each worker must exceed the public pension tax payment per worker after also rebating the increment to wages captured by the working cohort from the increased saving of the preceding cohort, now retired.[6] From equations (25) and (26), then,

$$\text{(27)} \quad \delta_{t+1} > \left[\frac{R_{t+1}}{G_{t+1}}\right] \delta_t > 0 .$$

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[6]Because each working cohort after the first receives higher wages under the compensating transfers program, the presence of the term $(w^o_{t+1} - w_{t+1})$ in equation (25) illustrates the insufficiency of the less demanding $\varphi_{t+1} > \tau w_{t+1}$ condition cited by Breyer as proof of the Pareto optimality of existing pay-as-you-go public pension programs. Equation (25) indicates that the increment to wages is insufficient to cover the difference between the compensating transfer payment and the public pension tax payment, a result not readily apparent in the Breyer derivation.

The interpretation of this result depends on the relationship between R and G. If the rate of return to capital always equals or exceeds the population growth rate under the pay-as-you-go public pension program ($R_t \geq G_t$), then equation (27) implies that the compensation pattern required to maintain the lifetime utility of each cohort is not sustainable over time, since the compensation difference per worker δ_t is unbounded above and will eventually outstrip the ability of workers to make the payment. Even if an ever increasing forced saving requirement were imposed on successive cohorts in an effort to raise the capital/labor ratio over time, diminishing returns would prevent output and wages from growing fast enough to forever sustain the required growth in the compensation payments.[7] At some future time, then, feasible compensation payments must fall short of the minimum required to maintain the lifetime utility of all affected cohorts at the same level as under the public pension program. The lifetime utility of at least some future cohorts must therefore be less under the compensating transfers program than under the public pension program, demonstrating that an existing pay-as-you-go public pension program is Pareto-superior to all compensating transfers schemes that require additional saving on the part of workers.

In the case where the rate of return to capital always falls short of the population growth rate under the pay-as-you-go public pension program ($R_t < G_t$), equation (27) indicates that the required compensation difference δ_t may diminish over time, although the compensation payment itself remains above the corresponding tax payment under the public pension program.[8] In

[7]Recall that the R_{t+1} appearing in equation (27) is the marginal product of capital under the previous pay-as-you-go public pension program and therefore is unaffected by forced saving increases under the new compensating transfers program.

[8]This follows directly from a rearrangement of equation (25).

contrast to the conclusion reached by Breyer, such compensation patterns may be feasible, implying the possible existence of compensation schemes that are Pareto-equivalent to an existing pay-as-you-go public pension program.

Under the assumptions of our model, however, this case is irrelevant in democratic societies with efficient political markets. Such societies would not rationally pursue investment to the extent that the rate of return to capital falls below the long-run population growth rate unless there were impediments to the initiation and expansion of a pay-as-you-go public pension program. Put another way, if such societies find themselves in the situation where the rate of return to capital always falls short of the population growth rate, the initiation or expansion of a pay-as-you-go public pension program can increase the lifetime income and consumption of all present and future cohorts and is therefore Pareto-superior to the status quo. Instead of reducing the size of an existing pay-as-you-go social security program, then, such societies should be expanding it.

In terms of the specifics of our model, the collective transformation frontier between consumption in the first and second periods of the life cycle for each cohort is initially determined by the production function, assuming that the net marginal product of the first unit of capital brought to production exceeds the population growth rate. As more first period consumption is deferred, the rate of return to capital falls until it equals the long-run population growth rate. At this point, the potential expansion of a pay-as-you-go public pension program, which allows intertemporal consumption transfers at the rate of growth in population, begins to define the collective consumption transformation frontier. Potential expansion of the public pension program continues to define the collective consumption transformation frontier for all further deferments of first period consumption, up to the point that first period consumption is forced to the minimum subsistence level. This point clearly occurs prior to the point at which $\tau = 1$, where public pension taxes completely exhaust first period income. At no point is the frontier defined by the domain of the production function for which the rate of return to capital lies below the long-run population growth rate, since expansion of the pay-as-you-go public pension program provides superior rates of intertemporal transformation to those provided by the expansion of saving in this domain of the production function.[9]

By simply setting the τ parameter to zero in the previous equations, these results can be extended to identify the Pareto characteristics of forced increases in saving in the context of an economy without an already existing public pension program. This extension leaves the conclusions reached above unaltered, with the implication that a forced increase in saving necessarily reduces the lifetime utility of at least one cohort in the relevant domain of the production function, whether or not a public pension program is in existence.

Under the assumptions of our model, then, these results suggest that any attempt to abolish or reduce in size an existing pay-as-you-go public pension program in a way that forces increases in saving is Pareto-inferior. More generally, these results suggest that any scheme incorporating forced increases in saving is Pareto-inferior, whether or not a public pension program is in place

[9]Under the assumptions of our model, this argument also suggests that we would only expect to find pay-as-you-go public pension programs in societies where the rate of return to capital and the population growth rate were equal. The more general condition that subsumes economies with positive productivity growth is the equality of the rate of return to capital and the aggregate economic growth rate. This conclusion does not necessarily extend to the real world, of course, because of complications not represented in our model, including uncertainty, the diversity of individual preferences, inefficient political markets, underemployment of resources, and non-constant population and productivity growth rates.

and whether or not compensation is attempted. In inefficient closed economies characterized by an economic growth rate in excess of the rate of return to capital, it may be possible to identify compensating transfers schemes incorporating forced increases in saving that are Pareto-equivalent to the status quo, in contrast to the conclusion reached by Breyer; if such schemes exist, however, they are Pareto-inferior to the initiation or expansion of a pay-as-you-go public pension program. In closed economies characterized by an economic growth rate equal to or below the rate of return to capital, compensating transfers schemes incorporating forced increases in saving are unsustainable and therefore Pareto-inferior to the status quo, necessarily reducing the lifetime utility of at least one present or future cohort. If forced increases in saving are not incorporated as part of the compensating transfers program, then compensation schemes can be found that are Pareto-equivalent to an existing pay-as-you-go public pension program, but such schemes are substantively equivalent to the public pension program and have no real economic effects on either the lifetime wealth or consumption of any cohorts or on the aggregate economy.

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